

Physics 128 Discussion (Week 3)

Problem 1 (The Planck Scale)

Suppose two nonrelativistic particles of charge e and mass M a distance r apart exert both gravitational and electrostatic forces on each other.

(a) Write the magnitude of the gravitational force between the two particles and the magnitude of the electrostatic force between the two particles in terms of e , M , r , and physical constants.

Newton's law gives the magnitude of the gravitational force

$$|\mathbf{F}_G| = G \frac{M^2}{r^2}, \quad (1)$$

and the magnitude of the electrostatic force is given by Coulomb's law

$$|\mathbf{F}_{EM}| = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2}. \quad (2)$$

(b) When M takes the value of the *Planck mass* M_{Pl} , the two forces you calculated in part (a) have approximately equal magnitudes. Find the Planck mass M_{Pl} in terms of Newton's constant G . You may assume that in natural units, the fine-structure constant $\alpha \equiv e^2/(4\pi\epsilon_0)$ is about $1/137$.

$|\mathbf{F}_G| = |\mathbf{F}_{EM}|$ is satisfied when $M = M_{\text{Pl}}$

$$GM_{\text{Pl}}^2 = \frac{e^2}{4\pi\epsilon_0} \equiv \alpha \approx \mathcal{O}(1). \quad (3)$$

Therefore, in terms of G , the Planck mass is given by

$$\begin{aligned} M_{\text{Pl}} &= \sqrt{\frac{1}{G}} \\ &\approx G^{-1/2}. \end{aligned} \quad (4)$$

(c) Using your answer from part (b), what units does G have in natural units? Confirm your answer by using the usual definition of G .

The fine-structure constant α is by definition the dimensionless quantity. Mass is measured in $[\text{GeV}]$ in natural units, then G has unit of inverse energy squared $[\text{GeV}^{-2}]$.

Conversely, the Planck mass has the same unit as $G^{-1/2}$. In natural unit

$$\begin{aligned} [G^{-1/2}] &= [G]^{-1/2} \\ &= \left[\frac{1}{\text{GeV}^2} \right]^{-1/2} \quad \left(\because |\mathbf{F}_G| = G \frac{M^2}{r^2} \right) \\ &= [\text{GeV}]. \end{aligned} \quad (5)$$

Therefore, $[M_{\text{Pl}}] = [\text{GeV}]$.

(d) The Planck mass is about 1.221×10^{19} GeV. Convert the Planck mass to SI units. Would you consider this an extreme mass value?

We need to convert $[\text{GeV}]$ in natural unit to kg in SI unit.

$$M_{\text{Pl}} = 1.221 \times 10^{19} \text{ GeV} \rightarrow M_{\text{Pl}}c^2 = 1.221 \times 10^{19} \text{ GeV}. \quad (6)$$

Then we get

$$\begin{aligned} M_{\text{Pl}} &= \frac{1.221 \times 10^{19} \text{ GeV}}{c^2} \\ &= \frac{1.221 \times 10^{19} \cdot 1.60218 \times 10^{-10} \text{ J}}{(3.0 \times 10^8 \text{ m/s})^2} \\ &\approx 2.18 \times 10^{-8} \text{ kg}. \end{aligned} \quad (7)$$

This is not particularly extreme value for a mass in our daily scale. But on the scale of elementary particles, M_{Pl} is extremely large.

(e) In quantum field theory, the forces in part (a) can be described using Feynman diagrams. Draw the simplest Feynman diagram for each of the processes that leads to the forces in part (a). (In quantum field theory, the *graviton* is the massless gauge boson that mediates gravitational interactions.) What do you think the coupling constants are in each case?

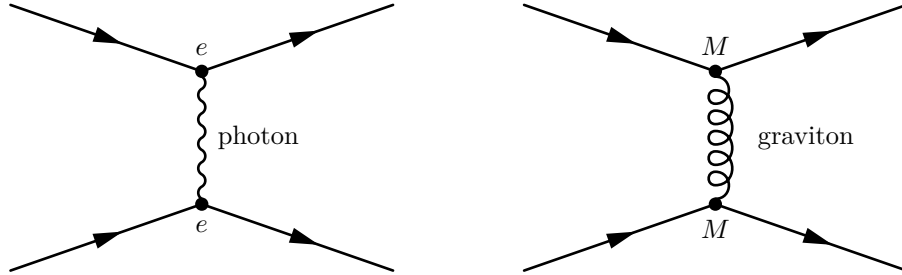


Figure 1: **(Left)** Schematic Feynman diagram of the electromagnetic interactions, which are mediated by photons. **(Right)** Schematic Feynman diagram of the gravitational interactions, which are mediated by hypothetical gravitons.

The Feynman diagrams for the simplest processes in electromagnetic and gravitational interactions are given in Fig. 1. For electromagnetism, the coupling constant should be related to the charge. The dimensionless coupling is

$$\sqrt{\alpha} = \frac{e}{\sqrt{4\pi\epsilon_0}} \quad (8)$$

For gravity, the coupling constant should be related to the mass M . To get a dimensionless coupling constant, we should divide M by the physical constant with units of mass: M_{Pl} . Then the full coupling constant for gravity is

$$\sqrt{G}M \approx \frac{M}{M_{\text{Pl}}}. \quad (9)$$

From this expression, you see that as M approaches M_{Pl} , gravity becomes strongly coupled.

Problem 2 (Muon Decay)

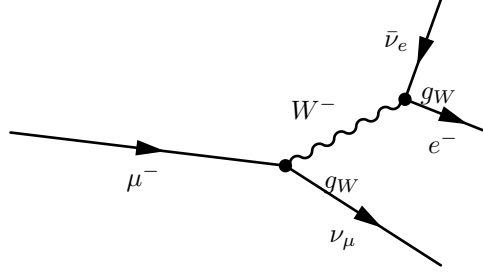


Figure 2: Muon decay mediated by W^- boson.

A muon (μ^- , rest mass about 105 MeV) at rest decays into a muon neutrino (ν_μ), an electron antineutrino ($\bar{\nu}_e$), and an electron e^- , via the exchange of a negatively charged W boson (W^- , rest mass about 80 GeV).

Neutrinos and antineutrinos are electrically neutral. We treat neutrinos, antineutrinos, and electrons as massless for the purposes of this problem. The Feynman diagram for this decay is shown in Fig. 3 (the arrows indicate the flow of fermion charge). The coupling constant for this weak interaction is g_W , which is unitless.

(a) Let $\vec{q}$ be the four-momentum of the W boson. Show that $\vec{q}^2 \ll M_W^2$ using conservation of energy and momentum.

Initially, the muon's four momentum is given by

$$\vec{p}_{\text{muon}} = \begin{pmatrix} m_\mu \\ \mathbf{0} \end{pmatrix}. \quad (10)$$

Let us set the direction of ν_μ along $\hat{z}$ direction, then the four momentum of ν_μ is given by

$$\vec{p}_{\nu_\mu} = \begin{pmatrix} E \\ E \end{pmatrix}, \quad (11)$$

here we used the fact that the neutrinos are set massless. The energy-momentum conservation reads

$$\vec{p}_{\text{muon}} = \vec{q} + \vec{p}_{\nu_\mu} \quad (12)$$

Thus W boson's four momentum $\vec{q}$ is given by

$$\begin{aligned} \vec{q} &= \vec{p}_{\text{muon}} - \vec{p}_{\nu_\mu} \\ &= \begin{pmatrix} m_\mu - E \\ -E \end{pmatrix} \end{aligned} \quad (13)$$

Since we ultimately produce an electron, the energy of the W boson must be at least the electron mass

$$\begin{aligned} m_\mu - E &\geq m_e \\ \Rightarrow E &\leq m_\mu - m_e \approx m_\mu. \quad (m_e \ll m_\mu : m_e \text{ is set massless here}) \end{aligned} \quad (14)$$

Now we have $E \leq m_\mu$. Thus

$$\begin{aligned} \vec{q}^2 &= (m_\mu - E)^2 - (-E)^2 \\ &= m_\mu^2 \left(1 - \frac{2E}{m_\mu} \right) \\ &\leq m_\mu^2 \\ &\ll M_W^2. \end{aligned} \quad (15)$$

Then we yield $\vec{q}^2 \ll M_W^2$.

(b) Approximate the amplitude of this Feynman diagram in terms of g_W and M_W .

The approximated scattering amplitude for this diagram is given by

$$\begin{aligned}\mathfrak{M} &= g_W^2 \left(\frac{1}{\vec{q}^2 - M_W^2} \right) \\ &\approx -\frac{g_W^2}{M_W^2} \quad (\because \vec{q}^2 \ll M_W^2).\end{aligned}\tag{16}$$

(c) Using dimensional analysis, transform your amplitude from part (b) into a decay rate. Express your answer in terms of g_W , M_W , and M_μ . Remember, in order to get a rate from an amplitude, you must start by squaring the amplitude. Note that all the particular details of the interaction should be included in the amplitude, so any corrections you make to the amplitude should be in terms of M_μ only. Scattering probability is proportional to the absolute square of the scattering amplitude.

$$\begin{aligned}\Gamma \propto |\mathfrak{M}|^2 &\approx \left| -\frac{g_W^2}{M_W^2} \right|^2 \\ &= \frac{g_W^4}{M_W^4}.\end{aligned}\tag{17}$$

From this, we see that $|\mathfrak{M}|^2$ has a unit of $[\text{GeV}^{-4}]$ in natural unit. We need to adjust the unit to make Γ have the unit of $[\text{GeV}]$ in the natural unit. Naively, we expect that Γ also depend on the muon mass m_μ . Then we adjust the unit of Γ by multiplying the power of m_μ

$$\Gamma \sim \frac{g_W^4}{M_W^4} m_\mu^5.\tag{18}$$

This is the approximated decay rate of muon. (Check: $[\Gamma] = [\text{GeV}^{-4}] [\text{GeV}]^5 = [\text{GeV}]$)

(d) Using dimensional analysis, transform your decay rate from part (d) into the *lifetime* of a muon at rest. Given that the measured lifetime of a muon at rest is about $2.2\mu\text{s}$, find an approximate value for g_W . In natural unit, the lifetime of muon is given by

$$\begin{aligned}\tau_\mu &= \frac{1}{\Gamma} \\ &\sim \frac{M_W^4}{g_W^4 m_\mu^5}.\end{aligned}\tag{19}$$

In SI unit, the measured lifetime of muon is about $2.2\mu\text{s}$. In natural unit, this value is given by

$$\begin{aligned}\tau_{\text{measured}} = 2.2\mu\text{s} &\rightarrow \tau_{\text{measured}} = \frac{2.2\mu\text{s}}{\hbar} \\ &= 3.348 \times 10^9 \text{ eV}^{-1} \\ &= 3.348 \times 10^{18} \text{ GeV}^{-1}.\end{aligned}\tag{20}$$

Then, comparing the measured value and the approximated form of the lifetime, we get

$$\begin{aligned}g_W^4 &= \frac{M_W^4}{m_\mu^5} \frac{1}{\tau_{\text{measured}}} \\ &= \frac{(80 \text{ GeV})^4}{(105 \text{ MeV})^5} \frac{1}{3.348 \times 10^{18} \text{ GeV}^{-1}} \\ &= \frac{(80 \text{ GeV})^4}{(0.105 \text{ GeV})^5} \frac{1}{3.348 \times 10^{18} \text{ GeV}^{-1}} \\ &\approx 0.959 \times 10^{-6}.\end{aligned}\tag{21}$$

Therefore you get

$$\begin{aligned}g_W &\approx (0.959 \times 10^{-6})^{1/4} \\ &\sim 0.03.\end{aligned}\tag{22}$$

Actual value of g_W is $g_W \sim 0.65$. This difference comes from the missing factor, which we neglect in the calculations to simplify the problem. Historically, we define the Fermi constant

$$G_F \equiv \left(\frac{g_W}{M_W} \right)^2, \quad (23)$$

which is the “effective” coupling constant we would have assumed if we did not know about the W boson and assumed that all four particles interacted at one vertex.

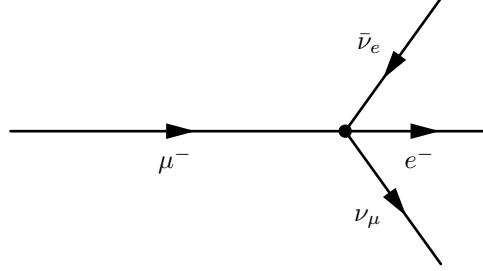


Figure 3: The Fermi's effective theory description of the Muon decay.

This description works well below 80 GeV, which is the mass of W. This effective description is invalid as you go to the higher and higher energy. This is the first example of the effective field theory in QFT: You don't have to deal with the subscale physics to get the valid result of a low energy scale.